

# A BICOMMUTANT THEOREM FOR DUAL BANACH ALGEBRAS

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## Abstract

A dual Banach algebra is a Banach algebra which is a dual space, with the multiplication being separately weak\*-continuous. We show that given a unital dual Banach algebra  $\mathcal{A}$ , we can find a reflexive Banach space  $E$ , and an isometric, weak\*-weak\*-continuous homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$  such that  $\pi(\mathcal{A})$  equals its own bicommutant.

*Keywords:* dual Banach algebra, bicommutant, reflexive Banach space.

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## 1 Introduction

Given a Banach space  $E$ , we write  $\mathcal{B}(E)$  for the Banach algebra of operators on  $E$ . Given a subset  $X \subseteq \mathcal{B}(E)$ , we write  $X'$  for the commutant of  $X$ ,

$$X' = \{T \in \mathcal{B}(E) : TS = ST \ (S \in X)\}.$$

The von Neumann bicommutant theorem tells us that if  $E$  is a Hilbert space, and  $X$  is a \*-closed, unital subalgebra, then  $X''$  is the strong operator topology closure of  $X$  in  $\mathcal{B}(E)$ . If  $X$  is not \*-closed, then this result may fail (consider strictly upper-triangular two-by-two matrices). However, a result of Blecher and Solel, [2], shows, in particular, that if  $X$  is weak\*-closed, that we can find another Hilbert space  $K$ , and a completely isometric, weak\*-weak\*-continuous homomorphism  $\pi : X \rightarrow \mathcal{B}(K)$ , such that  $\pi(X) = \pi(X)''$ . That is, if we change the Hilbert space which our algebra acts on, we do have a bicommutant theorem.

A dual Banach algebra is a Banach algebra which is a dual space, such that the multiplication is weak\*-continuous. Building on work of Young and Kaiser, the author showed in [5] that given a dual Banach algebra  $\mathcal{A}$ , we can find a reflexive Banach space  $E$  and an isometric, weak\*-weak\*-continuous homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$ . In this paper, we show that when  $\mathcal{A}$  is unital, we can choose  $E$  and  $\pi$  such that  $\pi(\mathcal{A}) = \pi(\mathcal{A})''$ . The method is similar to that used in [2] (although we follow the presentation of [1]) combined with an idea adapted from [5, Section 6].

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## 2 Notation and preliminary results

Given a Banach space  $E$ , let  $E^*$  be the dual space to  $E$ . For  $\mu \in E^*$  and  $x \in E$ , we write  $\langle \mu, x \rangle = \mu(x)$ . For  $X \subseteq E$ , let

$$X^\perp = \{\mu \in E^* : \langle \mu, x \rangle = 0 \ (x \in X)\}.$$

For  $Y \subseteq E^*$ , let

$${}^\perp Y = \{x \in E : \langle \mu, x \rangle = 0 \text{ } (\mu \in Y)\}.$$

Then  ${}^\perp(X^\perp)$  is the closure of the linear span of  $X$ , while  $({}^\perp Y)^\perp$  is the weak\*-closure of the linear span of  $Y$ . We may canonically identify  $X^*$  with  $E/X^\perp$ , and  $(E/X)^\perp$  with  $X^\perp$ . In particular,  $Y$  is weak\*-closed if and only if  $Y = ({}^\perp Y)^\perp$ , and in this case, the canonical predual of  $Y$  is  $E/{}^\perp Y$ .

We write  $E^* \widehat{\otimes} E$  for the projective tensor product of  $E^*$  with  $E$ . This is the completion of the algebraic tensor product  $E^* \otimes E$  with respect to the norm

$$\|\tau\|_\pi = \inf \left\{ \sum_{k=1}^n \|\mu_k\| \|x_k\| : \tau = \sum_{k=1}^n \mu_k \otimes x_k \right\}.$$

Any element of  $E^* \widehat{\otimes} E$  can be written as  $\sum_k \mu_k \otimes x_k$  with  $\sum_k \|\mu_k\| \|x_k\| < \infty$ . For further details, see [3] or [6], for example.

The Banach algebra  $\mathcal{B}(E)$  is a dual Banach algebra with respect to the predual  $E^* \widehat{\otimes} E$ , the dual pairing being given by

$$\langle T, \mu \otimes x \rangle = \langle \mu, T(x) \rangle \quad (T \in \mathcal{B}(E), \mu \otimes x \in E^* \widehat{\otimes} E),$$

and linearity and continuity. Indeed, under many circumstances, this is the unique predual for  $\mathcal{B}(E)$ , see [5, Theorem 4.4].

It follows that any weak\*-closed subalgebra of  $\mathcal{B}(E)$  is also a dual Banach algebra: then [5, Corollary 3.8] shows that every dual Banach algebra arises in this way. If  $X \subseteq \mathcal{B}(E)$ , then  $X'$  is a closed subalgebra of  $\mathcal{B}(E)$ . Notice that  $T \in X'$  if and only if  $T$  annihilates all  $\tau \in E^* \widehat{\otimes} E$  of the form

$$\tau = \mu \otimes S(x) - S^*(\mu) \otimes x \quad (S \in X, \mu \in E^*, x \in E).$$

Hence  $X' = Y^\perp = (E^* \widehat{\otimes} E/Y)^*$  is weak\*-closed, where  $Y$  is the closed linear span of such  $\tau$ . In particular,  $X''$  is a weak\*-closed subalgebra of  $\mathcal{B}(E)$  containing  $X$ , and so  $X''$  contains the weak\*-closed algebra generated by  $X$ .

We shall follow the ideas of [1, Theorem 3.2.14]; see [2] for a fuller treatment. We first establish some preliminary results. Given a Banach space  $E$ , we write  $\ell^2(E)$  for the Banach space consisting of sequences  $(x_n)$  in  $E$  with norm  $\|(x_n)\|_2 = \left( \sum_n \|x_n\|^2 \right)^{1/2}$ . Throughout, we could instead work with  $\ell^p(E)$  for  $1 < p < \infty$ , if we so wished. Then  $\ell^2(E)^* = \ell^2(E^*)$ , and  $\ell^2(E)$  is reflexive if  $E$  is. For each  $n$ , let  $\iota_n : E \rightarrow \ell^2(E)$  be the injection onto the  $n$ th co-ordinate, and let  $P_n : \ell^2(E) \rightarrow E$  be the projection onto the  $n$ th co-ordinate. For  $T \in \mathcal{B}(E)$ , let  $T^{(\infty)} \in \mathcal{B}(\ell^2(E))$  be the operator given by applying  $T$  to each co-ordinate. Notice that  $T^{(\infty)}\iota_n = \iota_n T$  and  $P_n T^{(\infty)} = T P_n$ , for each  $n$ . For  $X \subseteq \mathcal{B}(E)$ , let  $X^{(\infty)} = \{T^{(\infty)} : T \in X\}$ . Given a homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$ , let  $\pi^{(\infty)} : \mathcal{A} \rightarrow \mathcal{B}(\ell^2(E))$  by the homomorphism given by  $\pi^{(\infty)}(a) = \pi(a)^{(\infty)}$  for each  $a \in \mathcal{A}$ .

**Lemma 2.1.** *For a Banach space  $E$ , and  $X \subseteq \mathcal{B}(E)$ , we have that  $(X^{(\infty)})'' = (X'')^{(\infty)}$ .*

*Proof.* Let  $Q \in (X^{(\infty)})'$ . For  $n, m \in \mathbb{N}$  and  $S \in X$ , we have that  $P_n Q \iota_m S = P_n Q S^{(\infty)} \iota_m = P_n S^{(\infty)} Q \iota_m = S P_n Q \iota_m$ . Thus  $P_n Q^{(\infty)} \iota_m \in X'$ , for each  $n, m$ . Similarly, one can show that for  $Q \in \mathcal{B}(\ell^2(E))$ , if  $P_n Q \iota_m \in X'$  for all  $n, m$ , then  $Q \in (X^{(\infty)})'$ .

So, given  $T \in X''$  and  $Q \in (X^{(\infty)})'$ , we have that  $T P_n Q \iota_m = P_n Q \iota_m T$  for all  $n, m$ . Thus, for all  $n, m$ , it follows that  $P_n T^{(\infty)} Q \iota_m = P_n Q T^{(\infty)} \iota_m$ , from which it follows that  $T^{(\infty)} Q = Q T^{(\infty)}$ . Thus  $(X'')^{(\infty)} \subseteq (X^{(\infty)})''$ .

For the converse, let  $T \in (X^{(\infty)})''$ . For each  $n, m$ , notice that  $\iota_n P_m \in (X^{(\infty)})'$ , so that  $T \iota_n P_m = \iota_n P_m T$ . Let  $r \in \mathbb{N}$ , so that

$$T \iota_n \delta_{m,r} = T \iota_n P_m \iota_r = \iota_n P_m T \iota_r.$$

It follows that  $T\iota_r = \iota_r R$  for some  $R \in \mathcal{B}(E)$ , and that  $R$  does not depend upon  $r$ . Thus there must exist  $R \in \mathcal{B}(E)$  with  $T = R^{(\infty)}$ . Now let  $S \in X'$ , so that  $S^{(\infty)} \in (X^{(\infty)})'$ , and hence

$$(RS)^{(\infty)} = TS^{(\infty)} = S^{(\infty)}T = (SR)^{(\infty)}.$$

It follows that  $R \in X''$ , and hence that  $(X^{(\infty)})'' \subseteq (X'')^{(\infty)}$ .  $\square$

**Lemma 2.2.** *Let  $E$  be a reflexive Banach space, and let  $X \subseteq \mathcal{B}(E)$  be a subalgebra. Let  $X_w$  be the weak\*-closure of  $X$  in  $\mathcal{B}(E)$ , with respect to the predual  $E^* \widehat{\otimes} E$ . Then  $(X_w)^{(\infty)} = (X^{(\infty)})_w$ .*

*Proof.* Let  $T \in (X^{(\infty)})_w$ . For  $x \in E, \mu \in E^*$  and  $n \neq m$ , certainly  $\iota_n(\mu) \otimes \iota_m(x) \in {}^\perp(X^{(\infty)})$ , and so

$$0 = \langle \iota_n(\mu), T\iota_m(x) \rangle = \langle \mu, P_n T \iota_m(x) \rangle.$$

Thus  $P_n T \iota_m = 0$  whenever  $n \neq m$ . For any  $x, \mu, n$  and  $m$ , we also have that

$$\iota_n(\mu) \otimes \iota_n(x) - \iota_m(\mu) \otimes \iota_m(x) \in {}^\perp(X^{(\infty)}).$$

It follows that  $P_n T \iota_n = P_m T \iota_m$ . Combining these results, we conclude that  $T = S^{(\infty)}$  for some  $S \in \mathcal{B}(E)$ .

Let  $\tau \in {}^\perp X \subseteq E^* \widehat{\otimes} E$ , say  $\tau = \sum_k \mu_k \otimes x_k$ . For  $R \in X$  and each  $n$ , we have that

$$\langle R^{(\infty)}, \sum_k \iota_n(\mu_k) \otimes \iota_n(x_k) \rangle = 0,$$

so that  $\sigma = \sum_k \iota_n(\mu_k) \otimes \iota_n(x_k) \in {}^\perp(X^{(\infty)})$ . So

$$0 = \langle T, \sigma \rangle = \langle S^{(\infty)}, \sigma \rangle = \langle S, \tau \rangle,$$

from which it follows that  $S \in X_w$ . So  $(X^{(\infty)})_w \subseteq (X_w)^{(\infty)}$ .

For the converse, let  $T \in X_w$ , and let  $\tau \in {}^\perp(X^{(\infty)})$ , say  $\tau = \sum_n \mu_n \otimes x_n$ . By rescaling, we may suppose that  $\sum_n \|\mu_n\|^2 = \sum_n \|x_n\|^2 < \infty$ . For each  $n$ , we have that  $\mu_n = (\mu_k^{(n)})$ , say, where  $\|\mu_n\|^2 = \sum_k \|\mu_k^{(n)}\|^2$ . Thus  $\sum_{n,k} \|\mu_k^{(n)}\|^2 < \infty$ . Similarly, each  $x_n = (x_k^{(n)})$ , and  $\sum_{n,k} \|x_k^{(n)}\|^2 < \infty$ . We can now compute that, for  $S \in X$ ,

$$0 = \langle S^{(\infty)}, \tau \rangle = \sum_n \langle \mu_n, S^{(\infty)}(x_n) \rangle = \sum_{n,k} \langle \mu_k^{(n)}, S(x_k^{(n)}) \rangle,$$

so that  $\sigma = \sum_{n,k} \mu_k^{(n)} \otimes x_k^{(n)} \in {}^\perp X$  (where this sum converges absolutely by an application of the Cauchy-Schwarz inequality). Then  $0 = \langle T, \sigma \rangle = \langle T^{(\infty)}, \tau \rangle$ , from which it follows that  $T^{(\infty)} \in (X^{(\infty)})_w$ . So  $(X_w)^{(\infty)} \subseteq (X^{(\infty)})_w$ .  $\square$

The following lemma is usually stated in terms of “reflexivity” of a subspace of  $\mathcal{B}(E)$ , but this is a different meaning to that of a reflexive Banach space, so we avoid this terminology.

**Lemma 2.3.** *Let  $E$  be a reflexive Banach space, and let  $X \subseteq \mathcal{B}(E)$  be a weak\*-closed subspace. If  $T \in \mathcal{B}(\ell^2(E))$  is such that, for each  $x \in \ell^2(E)$ , we have that  $T(x)$  is in the closure of  $\{S^{(\infty)}(x) : S \in X\}$ , then actually  $T \in X^{(\infty)}$ .*

*Proof.* Let  $T$  be as stated, so for each  $n$ , we have that the image of  $T\iota_n$  is a subset of the image of  $\iota_n$ . By considering what  $T$  maps  $(\iota_1 + \cdots + \iota_n)(x)$  to, for any  $x \in E$ , we may conclude that  $T = R^{(\infty)}$  for some  $R \in \mathcal{B}(E)$ .

Let  $\tau \in {}^\perp X$ , say  $\tau = \sum_n \mu_n \otimes x_n$ , where we may suppose that  $\sum_n \|\mu_n\|^2 = \sum_n \|x_n\|^2 < \infty$ . Let  $\mu = (\mu_n) \in \ell^2(E^*)$  and  $x = (x_n) \in \ell^2(E)$ , so that

$$\langle R, \tau \rangle = \langle \mu, R^{(\infty)}(x) \rangle = \langle \mu, T(x) \rangle.$$

However, notice that  $\langle \mu, S^{(\infty)}(x) \rangle = \langle S, \tau \rangle = 0$  for each  $S \in X$ , so by the assumption on  $T$ , it follows also that  $\langle \mu, T(x) \rangle = 0$ , so  $\langle R, \tau \rangle = 0$ . So  $R \in ({}^\perp X)^\perp = X$ , that is,  $T \in X^{(\infty)}$ .  $\square$

### 3 The main result

Let us introduce some temporary terminology, motivated by [1]. Let  $\mathcal{A}$  be a Banach algebra, and  $E$  be a left  $\mathcal{A}$ -module (which we assume to be a Banach space with contractive actions). In this section, we shall always suppose that  $E$  is *essential*, that is, the linear span of  $\{a \cdot x : a \in \mathcal{A}, x \in E\}$  is dense in  $E$ .

We say that  $E$  is *cyclic* if there exists  $x \in E$  with  $\mathcal{A} \cdot x = \{a \cdot x : a \in \mathcal{A}\}$  being dense in  $E$ . We say that  $E$  is *self-generating* if, for each closed cyclic submodule  $K \subseteq E$ , the linear span of  $\{T(E) : T : E \rightarrow K \text{ is an } \mathcal{A}\text{-module homomorphism}\}$  is dense in  $K$ .

The following is very similar to the presentation in [1], but we check that the details still work for reflexive Banach spaces, and not just Hilbert spaces.

**Theorem 3.1.** *Let  $\mathcal{A}$  be a unital Banach algebra, and let  $E$  be a reflexive Banach space with a bounded homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$ . Use  $\pi$  to turn  $E$  into a left  $\mathcal{A}$ -module, and suppose that  $\ell^2(E)$  is self-generating. Then  $\pi(\mathcal{A})''$  agrees with the weak\*-closure of  $\pi(\mathcal{A})$  in  $\mathcal{B}(E)$ .*

*Proof.* Let  $\mathcal{B}$  be the closure of  $\pi(\mathcal{A})$  in  $\mathcal{B}(E)$ , and let  $\mathcal{B}_w$  be the weak\*-closure of  $\mathcal{B}$ . We wish to show that  $\mathcal{B}_w = \mathcal{B}''$ .

Let  $T \in (\mathcal{B}'')^{(\infty)} \subseteq \mathcal{B}(\ell^2(E))$ , let  $x \in \ell^2(E)$  be non-zero, and let  $K$  be the closure of  $\mathcal{B}^{(\infty)}(x)$ . As  $E$  is essential, it follows that the unit of  $\mathcal{A}$  acts as the identity on  $E$ , and hence also as the identity on  $\ell^2(E)$ , under  $\pi^{(\infty)}$ . Thus  $x \in K$ . We shall show that  $T(K) \subseteq K$ .

Let  $V : \ell^2(E) \rightarrow K$  be an  $\mathcal{A}$ -module homomorphism, and let  $\iota : K \rightarrow \ell^2(E)$  be the inclusion map. By continuity, and the density of  $\mathcal{A}$  in  $\mathcal{B}$ , we see that  $\iota V \in (\mathcal{B}^{(\infty)})'$ . Hence  $T\iota V = \iota VT$ , from which it follows that  $TV(\ell^2(E)) = VT(\ell^2(E)) \subseteq K$ . Let  $W$  be the linear span of the images of all such  $V$ . As  $\ell^2(E)$  is self-generating, it follows that  $W$  is dense in  $K$ . However,  $T(W) \subseteq K$ , and so by continuity,  $T(K) \subseteq K$ , as required.

So we have shown that for each  $x \in \ell^2(E)$ , we have that  $T(x)$  is in the closed linear span of  $\mathcal{B}^{(\infty)}(x) \subseteq \mathcal{B}_w^{(\infty)}(x)$ . By Lemma 2.3, we conclude that  $T \in \mathcal{B}_w^{(\infty)}$ . So we have shown that  $(\mathcal{B}'')^{(\infty)} \subseteq \mathcal{B}_w^{(\infty)}$ . By Lemma 2.1 and Lemma 2.2, this shows that  $(\mathcal{B}^{(\infty)})'' \subseteq (\mathcal{B}^{(\infty)})_w$ . However, we always have that  $(\mathcal{B}^{(\infty)})_w \subseteq (\mathcal{B}^{(\infty)})''$ , and so  $(\mathcal{B}^{(\infty)})_w = (\mathcal{B}^{(\infty)})''$ . Hence also  $(\mathcal{B}_w)^{(\infty)} = (\mathcal{B}'')^{(\infty)}$ , from which it follows immediately that  $\mathcal{B}_w = \mathcal{B}''$ , as required.  $\square$

By using the Cohen Factorisation theorem, see [3, Corollary 2.9.25], a slightly more subtle argument would show that this theorem also holds for Banach algebras with a bounded approximate identity.

The previous result is only useful if we have a good supply of self-generating modules. The following is similar to an idea we used in [5, Lemma 6.10].

**Proposition 3.2.** *Let  $\mathcal{A}$  be a Banach algebra, and let  $E$  be a reflexive Banach space which is a left  $\mathcal{A}$ -module. There exists a reflexive left  $\mathcal{A}$ -module  $F$  such that:*

1.  *$E$  is isomorphic to a one-complemented submodule of  $F$ ;*
2. *each closed, cyclic submodule of  $\ell^2(F)$  is isomorphic to a one-complemented submodule of  $F$ ;*

*In particular,  $\ell^2(F)$  is self-generating.*

*Proof.* Let  $\mathcal{E}_0 = \{E\}$ . We use transfinite induction to define  $\mathcal{E}_\alpha$  to be a set of reflexive left  $\mathcal{A}$ -modules, for each ordinal  $\alpha \leq \aleph_1$ . If  $\alpha$  is a limit ordinal, we simply define  $\mathcal{E}_\alpha = \bigcup_{\beta < \alpha} \mathcal{E}_\beta$ .

Otherwise, we let  $E_\alpha$  to be the  $\ell^2$  direct sum of each module in  $\mathcal{E}_\alpha$ , so that  $E_\alpha$  is a reflexive left  $\mathcal{A}$ -module in the obvious way. Let  $\mathcal{E}_{\alpha+1}$  be  $\mathcal{E}_\alpha$  unioned with the set of all closed cyclic submodules of  $\ell^2(E_\alpha)$ .

Let  $F$  be the  $\ell^2$  direct sum of all the modules in  $\mathcal{E}_{\aleph_1}$ . As  $\{E\} = \mathcal{E}_0 \subseteq \mathcal{E}_{\aleph_1}$ , condition (1) follows. Let  $K$  be a closed, cyclic submodule of  $\ell^2(F)$ , say  $K$  is the closure of  $\mathcal{A} \cdot x$ . Thus

$$x \in \ell^2(F) \cong \ell^2 - \bigoplus_{G \in \mathcal{E}_{\aleph_1}} \ell^2(G).$$

Say  $x = (x_G)_{G \in \mathcal{E}_{\aleph_1}}$  where each  $x_G \in \ell^2(G)$ . As  $\|x\|^2 = \sum_G \|x_G\|^2 < \infty$ , it follows that  $x_G \neq 0$  for at most countably many  $G$ . As  $\aleph_1$  is uncountable, we must actually have that there exists  $\alpha < \aleph_1$  with  $x \in \ell^2 - \bigoplus_{G \in \mathcal{E}_\alpha} \ell^2(G) \cong \ell^2(E_\alpha)$ . Then, by construction,  $K \in \mathcal{E}_{\alpha+1}$ , and so  $K$  is a one-complemented submodule of  $F$ .  $\square$

Let  $\mathcal{A}$  be a Banach algebra. Recall, for example from [5], that  $\text{WAP}(\mathcal{A}^*)$  is the closed submodule of  $\mathcal{A}^*$  consisting of those functionals  $\phi \in \mathcal{A}^*$  such that

$$\mathcal{A} \rightarrow \mathcal{A}^*; \quad a \mapsto a \cdot \phi$$

is weakly-compact. Young's result, [9], shows that for each  $\phi \in \text{WAP}(\mathcal{A}^*)$ , there exists a reflexive Banach space  $E$ , a contractive homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$ , and  $x \in E, \mu \in E^*$  with  $\|\phi\| = \|x\| \|\mu\|$  and such that

$$\langle \phi, a \rangle = \langle \mu, \pi(a)(x) \rangle \quad (a \in \mathcal{A}).$$

Let  $\mathcal{A}$  be a dual Banach algebra with predual  $\mathcal{A}_*$ . It is easy to show (see [5] for example) that  $\mathcal{A}_* \subseteq \text{WAP}(\mathcal{A}^*)$ . We showed in [5, Section 3] that Young's result holds for  $\phi \in \mathcal{A}_*$ , with the additional condition that for any  $\lambda \in E^*$  and  $y \in E$ , the functional  $\pi^*(\lambda \otimes y)$  is in  $\mathcal{A}_*$ , where

$$\langle \pi^*(\lambda \otimes y), a \rangle = \langle \lambda, \pi(a)(y) \rangle \quad (a \in \mathcal{A}).$$

Note that, a priori, Young's result only shows that  $\pi^*(\lambda \otimes y) \in \text{WAP}(\mathcal{A}^*)$ .

**Proposition 3.3.** *With the notation of Proposition 3.2, we have that  $\pi^*(F^* \widehat{\otimes} F)$  is a subset of the closed submodule generated by  $\pi^*(E^* \widehat{\otimes} E)$ .*

*Proof.* The module  $F$  is generated from  $E$  by two constructions: (i) taking submodules; and (ii) taking  $\ell^2$ -direct sums. For (i), let  $K$  be a submodule of  $E$ . The Hahn-Banach theorem shows that  $\pi^*(K^* \widehat{\otimes} K) \subseteq \pi^*(E^* \widehat{\otimes} E)$ . For (ii), let  $(K_i)$  be a family of submodules of  $E$  with  $\pi^*(K_i^* \widehat{\otimes} K_i) \subseteq \pi^*(E^* \widehat{\otimes} E)$  for each  $i$ , and let  $F = \ell^2 - \bigoplus_i K_i$ . Let  $\sum_n \mu_n \otimes x_n \in F^* \widehat{\otimes} F$ , with, say,  $\sum_n \|\mu_n\|^2 = \sum_n \|x_n\|^2 < \infty$ . For each  $n$ , we have  $\mu_n = (\mu_i^{(n)})$  with  $\|\mu_n\|^2 = \sum_i \|\mu_i^{(n)}\|^2$ , and  $x_n = (x_i^{(n)})$  with  $\|x_n\|^2 = \sum_i \|x_i^{(n)}\|^2$ . Then

$$\sum_n \langle \mu_n, a \cdot x_n \rangle = \sum_{n,i} \langle \mu_i^{(n)}, a \cdot x_i^{(n)} \rangle \quad (a \in \mathcal{A}).$$

Hence

$$\pi^*\left(\sum_n \mu_n \otimes x_n\right) = \pi^*\left(\sum_{n,i} \mu_i^{(n)} \otimes x_i^{(n)}\right) \in \pi^*(E^* \otimes E).$$

Again, the Cauchy-Schwarz inequality shows that the sum on the right converges.  $\square$

**Theorem 3.4.** *Let  $\mathcal{A}$  be a unital dual Banach algebra. There exists a reflexive Banach space  $E$  and an isometric, weak\*-weak\*-continuous homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$  such that  $\pi(\mathcal{A})'' = \pi(\mathcal{A})$ .*

*Proof.* By [5, Corollary 3.8], we may suppose that  $\mathcal{A} \subseteq \mathcal{B}(E_0)$ , for some reflexive Banach space  $E_0$ . By Proposition 3.2, we can find a self-generating, reflexive Banach space  $E$  and a contractive representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$ . As  $E_0 \subseteq E$ , it follows that  $\pi$  is an isometry. By Proposition 3.3,  $\pi$  is weak\*-weak\*-continuous. The result now follows from Theorem 3.1.  $\square$

It is well-known that for any Banach algebra  $\mathcal{A}$ , we have that  $\text{WAP}(\mathcal{A}^*)^*$  is a dual Banach algebra (see, for example, [5, Proposition 2.4]). When  $\mathcal{A}$  has a bounded approximate identity, a weak\*-limit point in  $\text{WAP}(\mathcal{A}^*)^*$  will be a unit for  $\text{WAP}(\mathcal{A}^*)^*$ .

**Corollary 3.5.** *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate identity. There exists a reflexive Banach space  $E$  and a contractive homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$  such that  $\pi(\mathcal{A})''$  is isometrically, weak\*-weak\*-continuously isomorphic to  $\text{WAP}(\mathcal{A}^*)^*$ .*

Finally, we remark that Uygul showed in [7] that given a dual, completely contractive Banach algebra  $\mathcal{A}$ , we can find a reflexive operator space and a completely isometric, weak\*-weak\*-continuous homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$ . Using this result, we can easily prove a version of Theorem 3.4 for completely contractive Banach algebras. Indeed, the only thing to do is to equip  $\ell^2$  direct sums with an Operator Space structure such that the inclusion and projection maps are complete contractions. This is worked out in detail in [8] (see also [7]).

Finally, we remark that the space constructed in Theorem 3.4 is very abstract. For a group measure space convolution algebra  $M(G)$ , Young showed in [9] that  $M(G)$  can be weak\*-represented on a direct sum of  $L^p(G)$  spaces; the analogous result for the Fourier algebra was shown by the author in [4]. For such concrete Banach algebras  $\mathcal{A}$ , it would be interesting to know if “nice” reflexive Banach spaces  $E$  could be found with  $\pi : \mathcal{A} \rightarrow \mathcal{B}(E)$  such that  $\pi(\mathcal{A})'' = \pi(\mathcal{A})$ .

## References

- [1] D. BLECHER, C. LE MERDY, *Operator Algebras and Their Modules: An Operator Space Approach*, (Clarendon Press, Oxford, 2004).
- [2] D. BLECHER, B. SOLEL, ‘A double commutant theorem for operator algebras’, *J. Operator Theory* 51 (2004) 435–453.
- [3] H. G. DALES, *Banach algebras and automatic continuity*, (Clarendon Press, Oxford, 2000).
- [4] M. DAWS, ‘Representing multipliers of the Fourier algebra on non-commutative  $L^p$  spaces’, preprint. See arXiv:0906.5128v2 [math.FA]
- [5] M. DAWS, ‘Dual Banach algebras: representations and injectivity’, *Studia Math.* 178 (2007) 231–275.
- [6] T. W. PALMER, *Banach algebras and the general theory of \*-algebras, Vol 1*, (Cambridge University Press, Cambridge, 1994).
- [7] F. UYGUL, ‘A representation theorem for completely contractive dual banach algebras’, *J. Operator Theory* 62 (2009) 327–340.
- [8] Q. XU, ‘Interpolation of operator spaces’, *J. Funct. Anal.* 139 (1996) 500–539.
- [9] N. J. YOUNG, ‘Periodicity of functionals and representations of normed algebras on reflexive spaces.’, *Proc. Edinburgh Math. Soc. (2)* 20 (1976/77) 99–120.

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